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Constructing and evaluating alternative frames of discernment ☆

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We construct alternative frames of discernment from input belief functions. We assume that the core of each belief function is a subset of a so far unconstructed frame of discernment. The alternative frames are constructed as different cross products of unions of different cores. With the frames constructed the belief functions are combined for each alternative frame. The appropriateness of each frame is evaluated in two ways: (i) we measure the aggregated uncertainty (an entropy measure) of the combined belief functions for that frame to find if the belief functions are interacting in interesting ways, (ii) we measure the conflict in Dempster's rule when combining the belief functions to make sure they do not exhibit too much internal conflict. A small frame typically yields a small aggregated uncertainty but a large conflict, and vice versa. The most appropriate frame of discernment is that which minimizes a probabilistic sum of the conflict and a normalized aggregated uncertainty of all combined belief functions for that frame of discernment.

1. Introduction

In this article we develop a problem representation that allows us to construct possible frames of discernment from a set of belief functions [2–6]. We assume that the core of each belief function is a subset of a so far unconstructed frame of discernment. The possible frames are constructed by partitioning the set of all cores into subsets. We continue by taking the union of each subset and then construct the possible frames by making cross products of these unions.

Each possible frame of discernment is evaluated on how well it yields focused and specific conclusions from the combination of the available belief functions without exhibiting too much internal conflict.

When conflict is higher than measurement errors it is a sign that something is wrong. It should be noted that there is at least two possible sources of conflict other than measurement errors. We may have modeling errors or faulty sources. Faulty sources are corrected by appropriate discounting (e.g. [7–9]) while modeling errors are corrected by adopting an appropriate frame of discernment. This article is concerned with constructing an appropriate frame of discernment. However, both approaches could be used either independently or together depending on the nature of the conflict. A classic modeling error was presented by Zadeh [10] where three non-exclusive diseases: meningitis (M), concussion (C) and brain tumor (T) were represented as atomic elements of a frame of discernment $\Theta = \{M, C, T\}$. This is in direct violation of the requirements of a frame of discernment to include only exclusive elements. Representing the problem with a frame like Θ means by definition that any patient must have exactly one disease, not zero, two or three multiple diseases. Such a modeling error can lead to

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counter-intuitive results when presented with highly conflicting evidence. A correction to the error was presented by Haenni [11] by adopting a frame of eight elements constructed as a cross product of the three elements of Θ, $\Theta' = \{\{M,NM\} \times \{C,NC\} \times \{T,NT\}\}$, where NM, NC and NT, means no M, no C and no T, respectively. This eliminates the problem and changes the conclusion dramatically.

Pal et al. investigated several different measures of uncertainty [12] and developed a new measure of average total uncertainty [13]. This is now superseded by the aggregated uncertainty (*AU*) functional [14–16] as the best generalized measure of both Shannon's entropy [17] and Hartley's [18] nonspecificity. In this article we develop a new method called frame appropriateness *FA*, where $1 - FA$ is the probabilistic sum of the conflict of Dempster's rule and *AU* normalized by $\log_2|\Theta|$. This method is used for evaluating the appropriateness of alternative frames of discernment where the *AU* part will be minimized (for small frames) to find interesting interactions among belief functions and the conflict part will be minimized (for large frames) to make sure the combination of belief functions does not exhibit too much internal conflict. Minimizing *FA* will typically yield a balance between the two parts of the measure.

The problem we are facing can thus be summarized as: we have some uncertain information about several different aspects of some phenomenon. We assume this information can be encoded as belief functions. However, we do not know the frame of discernment. Instead we try to construct the frame from the cores of the belief functions at hand. Here, we do not make any assumption that the cores are sets of atomic elements of the same frame as they may concern different aspects of the same phenomenon. Instead we assume that they may belong to different homogeneous subframes whose cross product is the frame representing all possibilities of the whole problem. This may however be revised whenever new information arrives and the frame might have to be expanded to include possible outcomes not known before.

As there may be several different alternative frames at any moment in time we want to find the most appropriate frame of discernment. We will define appropriateness of a frame of discernment in such a way as it fulfills two different aspects simultaneously. Shafer [5, p. 280], proposed that an ideal frame should simultaneously let our evidence "interact in an interesting way" without "exhibit too much internal conflict". We interpret "interesting" as having an as sharp distribution as possible, i.e., we like to see as much mass as possible distributed on as few and small focal elements as possible. Preferably, all mass focused on one focal element of cardinality one. The best way to measure how focused the distribution is on as few focal elements as possible is using the generalized Shannon entropy. The best way to measure how focused the distribution is on as small focal elements as possible is using the generalized Hartley information measure. Together they make up the aggregated uncertainty measure (*AU*). Finding a frame that minimizes *AU* for the combined distribution is our answer to finding the frame that best let our evidence "interact in an interesting way". At the same time we like the conflict of the combination of all belief functions to be as small as possible as any conflict larger than measuring errors is a sign that something (possibly the framing of the problem) is wrong. We measure the conflict of Dempster's rule when combining all belief functions on the frame, and like to see a conflict as low as possible. Finding a frame that minimizes the conflict is our answer to not "exhibit too much internal conflict".

Thus, we use *AU* as a penalty function and seek its minimum to have sharp distributions from which we can draw interesting conclusions and the conflict of the combination by Dempster's rule as a penalty function and seek its minimum to have distributions which we can trust. Finally, we want to see both considerations achieved simultaneously, i.e., both penalty functions minimized simultaneously in such a way that if one criteria fail their combination fail. This is achieved by minimizing their probabilistic sum.

Janez and Appriou [19] study methods for combining evidence represented on different homogeneous but non-exhaustive partial frames of discernment. They assume an open world where knowledge of the universe of possible outcomes is partial and where evidence from different sources may be defined on frames corresponding to different subsets of all possible outcomes. They propose and investigate a series of alternative methods of managing the situation where the simplest approach is based on deconditioning. Here, each focal element A is appended by union with all elements from other frames that are not also included in the frame that A is represented upon, before combination is performed, i.e., focal element A on frame Θ_i is substituted by

$$A \cup \left(\Theta_i^c \cap \bigcup_{j \neq i} \Theta_j \right), \tag{1}$$

where Θ_j, $i \neq j$, are the other frames. When additional information is available they propose methods that yield tighter results. In addition they develop two methods without deconditioning for the case with non-disjoint frames where it is only possible to calculate plausibilities. If evidence is known *a priori* to be homogeneous the deconditioning and union of frames proposed by Janez and Appriou may be used instead of taking cross products of subframes as described in this article.

In addition some ad hoc methods based on linear combination as a method of combining evidence from different frames without extensions of focal elements are discussed in [20].

In an article by Denœux and Yaghlane [21] the authors take an opposite approach. Instead of building a frame bottom-up from data they assume an existing frame of discernment and seek to reduce its size by sequential coarsening of pairs of elements of the frame. Selecting which two elements that are to be combined at each step is done be minimizing information loss. Reducing the size of the frame is done in order to improve on the computational complexity of combining mass functions by using Dempster's rule for commonalities. This approach is usually faster than direct application of Dempster's rule. When combining mass functions in this way it is necessary to first transform all mass functions into commonalities by the

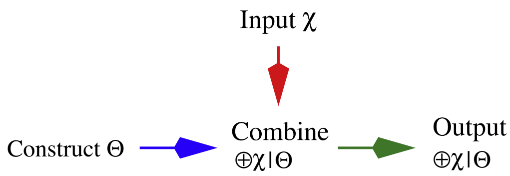

Fig. 1. A simplistic workflow model of constructing frames of discernment and combining evidence, where Θ is the frame of discernment, χ is the set of all belief functions and $\oplus$ is Dempster's rule.

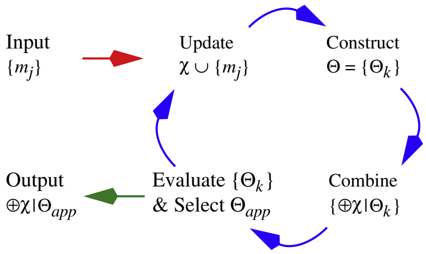

Fig. 2. An iterative workflow model of constructing frames of discernment and combining evidence, where $\{\Theta_k\}$ is a set of alternative frames of discernment, $\{m_j\}$ is a set of newly arriving additional belief functions and Θ_{app} is the most appropriate frame of discernment.

fast Möbius transform [22]. The computational complexity of this transformation depends exponentially on the size of the frame. While this approach is still exponential in the smaller frame the authors show in a benchmark test that they can reduce the computation time to 1/2000 with 99% accuracy.

Coarsening is different from the abridgement technique used in this article. A coarsed frame is an approximation with lower resolution of the same problem, while abridgment is a new frame for a new problem where some elements are eliminated from the frame. The methods described in [21] could however be used in a second step after the most appropriate frame of discernment has been constructed to yield faster computation in case a smaller approximate frame is needed for computational reasons.

Abellán and Moral [23] develop a new method for learning credal classification trees from data that improves upon earlier methods. While this is a different method for a different problem it has some similar characteristics with the methods developed in this article. These methods can be viewed as generalizations on Quinlan's classic ID3 learning algorithm [24]. One innovation is the use of a measure of total uncertainty with the usual entropy part and a nonspecificity part. They point out that using this measure they avoid the problem of over fitting data. The tree is build sequentially. If a branch is to be added to the tree depends on the change of total uncertainty. When a branch is added it may decrease the entropy part but increase the nonspecificity part of the total uncertainty measure. They use this measure as the stopping criterion, stopping the tree growth whenever the measure cannot decrease any further. This minimization of a measure with two parts balanced against each other (for selection between different classification trees) is similar to the minimization of *FA* where conflict and *AU* are balanced against each other (for selection between different frames of discernment).

With the methodology developed in this article we may work in a natural iterative way with the problem of frame construction and the problem of belief combination and systematically evaluate different possible frames. As we receive more evidence we will adjust our frame, possibly enlarging it from the previously one used. This changes probable reasoning from a linear approach of frame construction followed by belief combination (Fig. 1), into an update-construct-combine-evaluate loop approach, where we simultaneously reason about the framing of the problem at hand and the solution to this problem, as shown in Fig. 2.

The methodology for constructing a frame for *one* problem (in this article) can be extended into a methodology for constructing multiple frames from a set of belief functions for *several* different subproblems [25].

In Section 2 we investigate constructing frames of discernment from incoming belief functions. In Section 3 we develop a measure for evaluating each frame on the grounds of its dual appropriateness in facilitating interesting results from combination of all belief functions without too much internal conflict. This is what Shafer calls "the dual nature of probable reasoning" [5, Chapter 12]. In Section 4 we develop an algorithm for constructing an appropriate frame of discernment using the results of the previous two sections. Finally, in Section 5 conclusions are drawn.

2. Constructing frames of discernment

Assume we have a set of evidence

$$\chi = \{m_i\}, \tag{2}$$

that originates from *one* problem with yet undetermined representation, where the set of $\{m_i\}$ are all belief functions at hand. The focal elements of each belief function m_i contain pieces of that representation.

Our task is to find the most appropriate frame of discernment that lets our evidence "interact in an interesting way" without "exhibit too much internal conflict" in the words of Shafer [5, p. 280].

This will usually not be the union of all cores of m_i as different cores may hold non-exclusive elements. For example, one belief function may assign support to a focal element "Red" in relation to the color of a car. Another belief function may assign support to a focal element "Fast" in relation to speed of that car. Obviously, "Red" and "Fast" are not both elements of the frame of discernment as they are not exclusive. However, the "(Red, Fast)" pair might be an element of a frame.

Our task of finding the most appropriate frame of discernment becomes finding the most appropriate cross product of different unions of cores, where each core C_i of m_i is included in one of the unions exactly once.

We define the most appropriate frame of discernment as the cross product of different unions of cores that maximizes a measure of frame appropriateness (*FA*), equal to one minus the probabilistic sum of the conflict of Dempster's rule and a normalized aggregated uncertainty (*AU*) of all combined belief functions.

Let us begin by introducing the representation needed to induce a frame of discernment from input data. After which, we will give an example and demonstrate how the frame of discernment can be modified by abridgment or enlargement [5].

2.1. Representation of frames of discernment

Assume we have a set of evidence χ. We observe the core C_i of each available belief function m_i. We assume that the core of each belief function is a subset of exclusive but not exhaustive elements of a so far unconstructed frame of discernment.

2.1.1. The set of cores

Let

$$C = \{C_i\}, \tag{3}$$

be the set of all cores of χ, where C_i is the core of m_i, the ith piece of evidence.

We have

$$C_i = \bigcup_j \{A_j | m_i(A_j) > 0\}, \tag{4}$$

where A_j is a focal element of m_i.

2.1.2. Partitioning the set of cores

Let

$$\Omega = \{\Omega_k\}, \tag{5}$$

be the set of all possible set partitions of C (the set of all cores), where Ω_k is the kth possible partition of C. The number of partitions of C is called a Bell number,[1] $B_{|C|}$, where

$$\begin{aligned} B_n &= \sum_{k=0}^{n-1} B_k \binom{n-1}{k}, \\ B_0 &= 1. \end{aligned} \tag{6}$$

We have

$$\Omega_k = \{\omega_l\}, \tag{7}$$

where the ω_l's are disjoint subsets of C, i.e.,

$$\forall l.\ \omega_l \subseteq C, \tag{8}$$

such that

$$\bigcup_l \omega_l = \{C_i\} \equiv C \tag{9}$$

and

$$\omega_m \cap \omega_n = \emptyset, \tag{10}$$

whenever $m \neq n$.

[1] The first few Bell numbers are 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21,147, 115975.

2.1.3. Constructing frames from partitions of cores

Let

$$\Theta_k = \{\Theta_k\}, \tag{11}$$

be the set of all possible cross products relating to Ω, such that Θ_k is the cross product of all unions of elements of the partition Ω_k, Eq. (7).

We have

$$\Theta_k = \times\{\theta_l\}, \tag{12}$$

where θ_l is the union of the elements in ω_l, $\omega_l \in \Omega_k$, and θ_l must be an exclusive set of elements.

We have

$$\forall l.\ \theta_l = \bigcup \omega_l = \bigcup_i \{C_i | C_i \in \omega_l\}, \tag{13}$$

such that

$$\bigcup_l \theta_l = \bigcup_l \left\{\bigcup \omega_l\right\} = \bigcup_i \{C_i\} = \bigcup C, \tag{14}$$

where all θ_l's observe two different crucial type conditions:

Type condition 1. No element of any θ_p may belong to any other cross product elements θ_q, i.e.,

$$\theta_p \cap \theta_q = \emptyset, \tag{15}$$

whenever $p \neq q$.

This will eliminate any frame that obviously distributes elements of the same type over different cross product elements. It is possible to strengthen type condition 1 further by going beyond checking intersections and doing type control between all pairs of cross product elements. This, however, is outside the scope of this article as it can not be decided within the field of statistics, i.e., there is no way within statistics to decide if "Fast" and "Red" are exclusive elements.

Type condition 2. Every cross product element θ_l must be an exclusive set, i.e.,

$$e_m \cap e_n = \emptyset, \tag{16}$$

whenever

$$\forall m, n \exists l. \quad e_m, e_n \in \theta_l. \tag{17}$$

As with type condition 1, the exclusivity of θ_l must be verified by methods outside of statistics, and thus, outside the scope of this article. The obvious solution to automate this verification is to have all items in our known universe of cars classified as to their color and speed.

We deliberately choose to cast type condition 2 in terms of exclusivity for two reasons. First, this is done to be fully consistent with the exclusivity requirement of a frame of discernment when the cross product is specialized to a single cross product element, i.e., a frame of discernment of atoms. Secondly, as with type condition 1, going one step further by asking that all elements of each cross product element is of the same type is very difficult and not within statistics. It may not even be preferable. Describing cars as 'Red' or 'Fast' within one cross product element (or within a frame of atoms without cross products) may be preferable if we know that there are no cars that are both 'Red' and 'Fast', i.e., the intersection between all items of the sets labeled 'Red' and 'Fast', respectively, is empty, Red $\cap$ Fast = $\emptyset$.

The necessity of exclusivity among elements of the cross product elements should be noted. Without exclusivity we have no conflict which is of crucial importance as the measure of frame appropriateness *FA* is based on a balance between conflict and entropy where conflict tends to increase with small frames and entropy increases with large frames.

The Θ_k's constructed where all θ_l meet exclusivity are the alternative frames of discernment. Our task is to find the most appropriate frame that let our evidence "interact in an interesting way" without "exhibit too much internal conflict". This will be examined in Section 3.

2.2. An example

Let us assume we have three belief functions available and we want to construct all alternative frames of discernment.

2.2.1. The set of cores

The set of evidence of the three belief functions is $\chi = \{m_1, m_2, m_3\}$ with

$$m_1 : \quad \begin{aligned} &\{[A_{11}, m_1(A_{11}) = 0.4] \\ &[A_{12}, m_1(A_{12}) = 0.4] \\ &[A_{13}, m_1(A_{13}) = 0.2]\}, \end{aligned} \tag{18}$$

where $F_1 = \{A_{11}, A_{12}, A_{13}\}$ is the set of focal elements of m_1. Assume that

$$\begin{aligned} A_{11} &= \{\text{Red}, \text{Green}\}, \\ A_{12} &= \{\text{Red}, \text{Blue}\}, \\ A_{13} &= \{\text{Red}\}. \end{aligned} \tag{19}$$

We find the core of m_1 using Eq. (4),

$$C_1 = \bigcup_j A_{1j} = \{\text{Red}, \text{Green}, \text{Blue}\}. \tag{20}$$

Furthermore, assume

$$m_2 : \quad \begin{aligned} &\{[\{\text{Fast}, \text{VeryFast}\}, m_2(\{\text{Fast}, \text{VeryFast}\}) = 0.8] \\ &[\{\text{Fast}\}, m_2(\{\text{Fast}\}) = 0.2]\} \end{aligned} \tag{21}$$

and

$$m_3 : \quad \begin{aligned} &\{[\{\text{Red}, \text{Black}\}, m_3(\{\text{Red}, \text{Black}\}) = 0.3] \\ &[\{\text{Red}\}, m_3(\{\text{Red}\}) = 0.7]\} \end{aligned} \tag{22}$$

with $C_2 = \{\text{Fast}, \text{VeryFast}\}$ and $C_3 = \{\text{Red}, \text{Black}\}$, respectively, where $C = \{C_1, C_2, C_3\}$ is the set of all cores of χ.

2.2.2. Partitioning the set of cores

The set of all cores C can be partitioned in five different ways.

We have a set of all possible partitions $\Omega = \{\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5\}$ of C where

$$\begin{aligned} \Omega_1 &= \{\omega_{11}, \omega_{12}, \omega_{13}\}, \\ \Omega_2 &= \{\omega_{21}, \omega_{22}\}, \\ \Omega_3 &= \{\omega_{31}, \omega_{32}\}, \\ \Omega_4 &= \{\omega_{41}, \omega_{42}\}, \\ \Omega_5 &= \{\omega_{51}\} \end{aligned} \tag{23}$$

with

$$\begin{aligned} &\omega_{11} = \{C_1\}, && \omega_{12} = \{C_2\}, \quad \omega_{13} = \{C_3\}, \\ &\omega_{21} = \{C_1, C_2\}, && \omega_{22} = \{C_3\}, \\ &\omega_{31} = \{C_1, C_3\}, && \omega_{32} = \{C_2\} \\ &\omega_{41} = \{C_2, C_3\}, && \omega_{42} = \{C_1\}, \\ &\omega_{51} = \{C_1, C_2, C_3\}. \end{aligned} \tag{24}$$

2.2.3. Constructing frames from partitions of cores

From $\Omega = \{\Omega_k\}$ we construct the set of all possible frames of discernment $\Theta = \{\Theta_k\}$ where each Θ_k corresponds to Ω_k. Using Eqs. (11) and (12) we obtain $\Theta = \{\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5\}$ where

$$\begin{aligned} \Theta_1 &= \theta_{11} \times \theta_{12} \times \theta_{13}, \\ \Theta_2 &= \theta_{21} \times \theta_{22}, \\ \Theta_3 &= \theta_{31} \times \theta_{32}, \\ \Theta_4 &= \theta_{41} \times \theta_{42}, \\ \Theta_5 &= \theta_{51} \end{aligned} \tag{25}$$

with, using Eq. (13),

$$\begin{aligned}
\Theta_1:\quad & \theta_{11} = C_1 = \{\mathrm{Red}, \mathrm{Green}, \mathrm{Blue}\},\\
& \theta_{12} = C_2 = \{\mathrm{Fast}, \mathrm{VeryFast}\},\\
& \theta_{13} = C_3 = \{\mathrm{Red}, \mathrm{Black}\},\\
\Theta_2:\quad & \theta_{21} = C_1 \cup C_2 = \{\mathrm{Red}, \mathrm{Green}, \mathrm{Blue}, \mathrm{Fast}, \mathrm{VeryFast}\},\\
& \theta_{22} = C_3 = \{\mathrm{Red}, \mathrm{Black}\},\\
\Theta_3:\quad & \theta_{31} = C_1 \cup C_3 = \{\mathrm{Red}, \mathrm{Green}, \mathrm{Blue}, \mathrm{Black}\},\\
& \theta_{32} = C_2 = \{\mathrm{Fast}, \mathrm{VeryFast}\},\\
\Theta_4:\quad & \theta_{41} = C_2 \cup C_3 = \{\mathrm{Red}, \mathrm{Black}, \mathrm{Fast}, \mathrm{VeryFast}\},\\
& \theta_{42} = C_1 = \{\mathrm{Red}, \mathrm{Green}, \mathrm{Blue}\},\\
\Theta_5:\quad & \theta_{51} = C_1 \cup C_2 \cup C_3 = \{\mathrm{Red}, \mathrm{Green}, \mathrm{Blue}, \mathrm{Black}, \mathrm{Fast}, \mathrm{VeryFast}\}.
\end{aligned} \tag{26}$$

However, Θ_1, Θ_2 and Θ_4 violate type condition 1, Eq. (15), and are not allowed as frames. This is determined by verifying that some intersections between different θ_l's for the same frame Θ_k are non-empty. For example,

$$\begin{array}{ll}
\Theta_1: & \theta_{11} \cap \theta_{13} = \{\mathrm{Red}\} \neq \emptyset\\
\Theta_2: & \theta_{21} \cap \theta_{22} = \{\mathrm{Red}\} \neq \emptyset\\
\Theta_3: & \theta_{41} \cap \theta_{42} = \{\mathrm{Red}\} \neq \emptyset.
\end{array} \tag{27}$$

Furthermore, Θ_2, Θ_4 and Θ_5 presumably violate the exclusivity condition of Eq. (16) and are not allowed as frames. We have

$$\begin{array}{ll}
\Theta_2, \theta_{21}: & \forall(i = 1,2,3)\ \forall(j = 4,5).\quad e_{21i} \cap e_{21j} \neq \emptyset\\
\Theta_4, \theta_{41}: & \forall(i = 1,2)\ \forall(j = 3,4).\quad e_{21i} \cap e_{21j} \neq \emptyset\\
\Theta_5, \theta_{51}: & \forall(i = 1,2,3,4)\ \forall(j = 5,6).\quad e_{21i} \cap e_{21j} \neq \emptyset.
\end{array} \tag{28}$$

For example, e_{211} and e_{214} of θ_{21},

$$e_{211} \cap e_{214} = \mathrm{Red} \cap \mathrm{Fast} \neq \emptyset, \tag{29}$$

are presumable non-exclusive elements and cannot both be elements of the same frame of discernment, although it may well be that the pair "(Red, Fast)" is an element of a frame. That something may be both "Red" and "Fast" making these elements non-exclusive must be established by other means.

From this frame construction process only Θ_3 comes through a possible frame of discernment. We have

$$\begin{aligned}
\Theta_3 &= \theta_{31} \times \theta_{32} = \left(\bigcup \omega_{31}\right) \times \left(\bigcup \omega_{32}\right) = \left(\bigcup\{C_1, C_3\}\right) \times \left(\bigcup\{C_2\}\right) = (C_1 \cup C_3) \times (C_2)\\
&= (\{\mathrm{Red}, \mathrm{Green}, \mathrm{Blue}\} \cup \{\mathrm{Red}, \mathrm{Black}\}) \times (\{\mathrm{Fast}, \mathrm{VeryFast}\})\\
&= \{\mathrm{Red}, \mathrm{Green}, \mathrm{Blue}, \mathrm{Black}\} \times \{\mathrm{Fast}, \mathrm{VeryFast}\}\\
&= \{(\mathrm{Red}, \mathrm{Fast}), (\mathrm{Red}, \mathrm{VeryFast}), (\mathrm{Green}, \mathrm{Fast}), (\mathrm{Green}, \mathrm{VeryFast}),\\
&\qquad (\mathrm{Blue}, \mathrm{Fast}), (\mathrm{Blue}, \mathrm{VeryFast}), (\mathrm{Black}, \mathrm{Fast}), (\mathrm{Black}, \mathrm{VeryFast})\}.
\end{aligned} \tag{30}$$

2.2.4. Reformulating belief functions given constructed frames

The one remaining thing to do is to reformulate our three belief functions given Θ_3. We get $\chi' = \{m'_1, m'_2, m'_3\}$ with

$$m'_1: \quad \begin{array}{l}
\{[A_{11}, m_1(A_{11}) = 0.4]\\
[A_{12}, m_1(A_{12}) = 0.4]\\
[A_{13}, m_1(A_{13}) = 0.2]\}
\end{array} \tag{31}$$

where

$$\begin{aligned}
A_{11} &= \{(\mathrm{Red}, \mathrm{Fast}), (\mathrm{Red}, \mathrm{VeryFast}), (\mathrm{Green}, \mathrm{Fast}), (\mathrm{Green}, \mathrm{VeryFast})\},\\
A_{12} &= \{(\mathrm{Red}, \mathrm{Fast}), (\mathrm{Red}, \mathrm{VeryFast}), (\mathrm{Blue}, \mathrm{Fast}), (\mathrm{Blue}, \mathrm{VeryFast})\},\\
A_{13} &= \{(\mathrm{Red}, \mathrm{Fast}), (\mathrm{Red}, \mathrm{VeryFast})\}
\end{aligned} \tag{32}$$

and similarly for the two remaining belief functions m'_2 and m'_3.

Thus, we have successfully constructed a frame of discernment Θ_3 from a set χ of three input belief functions. Using this frame we have reformulated the three belief functions in the terms of the adopted frame.

2.3. Abridgment

For all possible frames of discernment $\{\Theta_k\}$, where $|\Theta_k| > 1$, we may include further assumptions that make the frames tighter. This may lead to more interesting interaction between the belief functions and lead to firmer conclusions provided that the conflict does not increase in any significant way. Every frame is based on assumptions. The frame we begin with is based on the assumption that the elements of that frame are all disjunct possible alternatives, and that no other possibilities exist. Whether a tighter or looser frame is to be preferred is a matter of appropriateness. Most often this will be a point of balance where meaningful interaction is weighted against too much conflict.

Let us study one particular frame of discernment Θ_i from the remaining set of possible frames Θ that observe both type condition 1 and 2, Eqs. (15) and (16), respectively. We have

$$\Theta_i = \times\{\theta_l\}. \tag{33}$$

For each cross product element there are $|2^{\theta_l}| - 2$ possible abridgments as each cross product element θ_l may be replaced by any smaller element of its own power set, except $\emptyset$. At least one cross product element θ_l must be abridged to construct a new abridged frame of Θ_i. We have a set of all possible abridgments of Θ_i,

$$\Theta'_i = \left\{\Theta'_{ij}\right\}_j = \left\{\times\left\{\theta'_{lj}\right\}\right\}_j, \tag{34}$$

where

$$\theta'_{lj} \in 2^{\theta_l} \tag{35}$$

and 2^{θ_l} is the power set of θ_l, $\theta'_{lj} \neq \emptyset$, and $\exists j.\ \theta'_{lj} \neq \theta_l$.

Thus, the set of all possible abridgments Θ'_i in addition to Θ_i itself, are possible frames of discernment that need to be evaluated for appropriateness.

When an abridgment is adopted as an alternative frame of discernment we must adjust all belief functions accordingly before combination. This is a simple two step process: First, all elements that are excluded from each θ'_{lj} must be eliminated from all focal elements of all belief functions. Secondly, if any focal element is reduced to $\emptyset$ its mass is redistributed proportionally to the remaining focal elements. If all focal elements are reduced to $\emptyset$ then the abridgment is impossible.

For the jth possible abridgment Θ'_{ij} of Θ'_i, we have

$$\begin{aligned} m''_p\left(A_q \cap \Theta'_{ij}\right) &= \frac{1}{1-k} \cdot m'_p(A_q), \quad A_q \cap \Theta'_{ij} \neq \emptyset, \\ m''_p(\emptyset) &= 0, \end{aligned} \tag{36}$$

where

$$k = \sum_p m'_p(A_q), \quad A_q \cap \Theta'_{ij} = \emptyset. \tag{37}$$

It is also possible to define m''_p without normalization, in Eq. (36). This approach will increase the ability to differentiate between different frames especially when conflict is absent.

2.3.1. The example

In Section 2.2 we studied an example and found a possible frame of discernment

$$\Theta_3 = \theta_{31} \times \theta_{32} = \{\text{Red}, \text{Green}, \text{Blue}, \text{Black}\} \times \{\text{Fast}, \text{VeryFast}\}. \tag{38}$$

From Θ_3 we may construct several different abridgments, where Θ_3 may be replaced by

$$\theta'_{31} \in 2^{\theta_{31}} \tag{39}$$

and

$$\theta'_{32} \in 2^{\theta_{32}}, \tag{40}$$

respectively, where $\theta'_{31}, \ \theta'_{32} \neq \emptyset$. Except that not both $\theta'_{31} = \theta_{31}$ and $\theta'_{32} = \theta_{32}$ are allowed.

As $|\theta_{31}| = 4$ and $|\theta_{32}| = 2$ we have $\left|\{\theta'_{31}\}\right| = 15$ and $\left|\{\theta'_{32}\}\right| = 3$. Thus, the number of possible abridgments to Θ_3 is 44 $\left(= \left|\{\theta'_{31}\}\right| \cdot \left|\{\theta'_{32}\}\right| - 1 = 15 \cdot 3 - 1\right)$.

When an abridged frame is adopted, all belief functions must be reformulated to eliminate those elements that do not belong to the new frame. For example, if θ_{31} is replaced by $\theta'_{314} = \{\text{Green}, \text{Blue}, \text{Black}\}$ excluding "Red" from θ_{31} we must reformulate m'_1 as

$$m''_1: \quad \begin{aligned} &\{[A_{11}, m_1(A_{11}) = 0.5], \\ &[A_{12}, m_1(A_{12}) = 0.5]\}, \end{aligned} \tag{41}$$

where

$$A_{11} = \{(\text{Green}, \text{Fast}), (\text{Green}, \text{VeryFast})\},$$
$$A_{12} = \{(\text{Blue}, \text{Fast}), (\text{Blue}, \text{VeryFast})\} \tag{42}$$

and similarly for m_2'' and m_3''.

2.4. Enlargement

We may make enlargements to any frame of discernment in the set of all constructed frames $\Theta = \{\Theta_k\}$. As the frames are constructed from available input belief functions, using all elements that appear in those belief functions, we do not have any further specific elements that are not already included in the frames. The only form of enlargement we can perform is to enlarge a particular cross product element θ_l with an element of unstated meaning. Let us denote these elements Λ_l, one for each θ_l.

Let us again take a look at frame Θ_i. We have

$$\Theta_i = \times\{\theta_l\}. \tag{43}$$

For each cross product element θ_l there is one possible enlargement: enlarging θ_l by Λ_l. At least one cross product element θ_l must be enlarged to construct a new enlarged frame of Θ_i. The set of all possible enlargements of Θ_i becomes

$$\Theta_i'' = \left\{\Theta_{lj}''\right\}_j = \left\{\times\left\{\theta_{lj}''\right\}\right\}_j, \tag{44}$$

where

$$\theta_{lj}'' \in \{\theta_{l'}, \theta_l + \{\Lambda_l\}\} \tag{45}$$

and $\exists j \cdot \theta_{lj}'' \neq \theta_l$. We have $2^{|\{\theta_l\}|} - 1$ possible enlargements of Θ_i, as each cross product element may or may not be enlarged.

Enlarging frames of discernment in this manner will partially remove any conflict within the cross product element where Λ_l is included. Including Λ_l in every θ_l will eliminate all conflict.

Thus, the set of all possible enlargements Θ_i'' are possible frames that need to be evaluated for appropriateness.

2.4.1. The example

We return to the example of Section 2.2 and the frame

$$\Theta_3 = \theta_{31} \times \theta_{32} = \{\text{Red}, \text{Green}, \text{Blue}, \text{Black}\} \times \{\text{Fast}, \text{VeryFast}\}. \tag{46}$$

We may construct three different enlargements of Θ_3, where θ_{31} and θ_{32} may be replaced by

$$\theta_{31}'' = \theta_{31} + \{\Lambda_{31}\} \tag{47}$$

and

$$\theta_{32}'' = \theta_{32} + \{\Lambda_{32}\}, \tag{48}$$

respectively. Except that not both $\theta_{31}'' = \theta_{31}$ and $\theta_{32}'' = \theta_{32}$ are allowed.

If, for example, θ_{32} is replaced by $\theta_{321}'' = \{\text{Fast}, \text{VeryFast}, \Lambda_{32}\}$ we must reformulate m_1' as

$$m_1''' : \{[A_{11}, m_1(A_{11}) = 0.4] \quad [A_{12}, m_1(A_{12}) = 0.4] \quad [A_{13}, m_1(A_{13}) = 0.2]\}, \tag{49}$$

where

$$A_{11} = \{(\text{Red}, \text{Fast}), (\text{Red}, \text{VeryFast}), (\text{Red}, \Lambda_{32}), (\text{Green}, \text{Fast}), (\text{Green}, \text{VeryFast}), (\text{Green}, \Lambda_{32})\},$$
$$A_{12} = \{(\text{Red}, \text{Fast}), (\text{Red}, \text{VeryFast}), (\text{Red}, \Lambda_{32}), (\text{Blue}, \text{Fast}), (\text{Blue}, \text{VeryFast}), (\text{Blue}, \Lambda_{32})\}, \tag{50}$$
$$A_{13} = \{(\text{Red}, \text{Fast}), (\text{Red}, \text{VeryFast}), (\text{Red}, \Lambda_{32})\}$$

and similarly for m_2''' and m_3'''.

3. Appropriate representation

In this section we will study how to evaluate the alternative frames of discernment on the grounds of being appropriate for yielding interesting interactions among the available belief functions without exhibiting too much internal conflict.

We will develop an overall measure of frame appropriateness FA that takes both considerations into account simultaneously. This measure must be a function of two other measures:

- one that directly measures the conflict in Dempster's rule when combining the belief functions [7], to make sure they do not exhibit too much internal conflict,
- another that measures the degree of interesting interaction among the belief functions by means of measuring how focused and specific the conclusions are that may be drawn from their combination.

We prefer to see basic belief masses that are focused on as few and as small focal elements as possible. This can be measured by generalizing Shannon's entropy [17] and Hartley's information [18] measures, respectively. We will use a measure of aggregated uncertainty that takes both types of uncertainty into account.

A small frame typically yields a small aggregated uncertainty but a large conflict, and vice versa. The most appropriate frame of discernment is that which finds a good balance between the two measures by maximizing the frame appropriateness *FA*.

Definition 1. Let Θ_k be a frame of discernment and let $\{m_j\}$ be a set of all available belief functions defined on Θ_k. We define a measure of frame appropriateness of Θ_k, denoted as $FA(\Theta_k|\{m_j\})$, by

$$FA(\Theta_k|\{m_j\}) = \begin{cases} [1 - Con(\oplus\{m_j|\Theta_k\})]\left[1 - \frac{AU(\oplus\{m_j|\Theta_k\})}{\log_2|\Theta_k|}\right], & |\Theta_k| > 1, \\ 1 - Con(\oplus\{m_j|\Theta_k\}), & |\Theta_k| = 1, \end{cases} \tag{51}$$

where *Con* is the conflict in Dempster's rule and *AU* is the functional called the aggregated uncertainty. We have $Con \in [0,1]$, $AU \in [0, \log_2|\Theta_k|]$ and $FA \in [0,1]$.

The measure of frame appropriateness *FA* is identical to one minus the probabilistic sum of conflict and normalized aggregated uncertainty.

The aggregated uncertainty functional *AU* is defined as

$$AU(\mathrm{Bel}) = \max_{\{p_x\}_{x\in\Theta}} \left\{ -\sum_{x\in\Theta} p(x)\log_2 p(x) \right\}, \tag{52}$$

where $\{p_x\}_{x\in\Theta}$ is the set of all probability distributions such that $p_x \in [0,1]$ for all $x \in \Theta$,

$$\sum_{x\in\Theta} p(x) = 1 \tag{53}$$

Input: a set of belief functions χ.

Output: Possible frames of discernment $\{\Theta_i\}$, $\{\Theta'_{ij}\}$, $\{\Theta''_{ij}\}$. Frame appropriateness $\forall ij.\ FA(\Theta_i|\chi)$, $FA(\Theta'_{ij}|\chi)$, $FA(\Theta''_{ij}|\chi)$.

Step 1. $\forall i.$ generate C_i using Eq. (4). Set $C = \{C_i\}$.

Step 2. $\forall k.$ generate Ω_k using Eq. (7)–Eq. (10). Set $\Omega = \{\Omega_k\}$.

Step 3. $\forall k.$ generate Θ_k using Eq. (11)–Eq. (12). Set $\Theta = \{\Theta_k\}$.

Step 4. $\forall ij.$ generate $\{\Theta'_{ij} \mid \forall kl. Con(\oplus\{m_j|\Theta'_{kl}\}) < 1,\ \Theta'_{kl} \supset \Theta'_{ij}\}_j$ using Eq. (34)–Eq. (35).

Step 5. $\forall k.$ If $Con(\oplus\{m_j|\Theta_k\}) > 0$ then $\forall j.$ generate Θ''_{ij}. Set $\Theta''_i = \{\Theta''_{ij}\}_j$.

Step 6. Compute evaluations of frame appropriateness $\forall ij.\ FA(\Theta_i|\chi)$, $FA(\Theta'_{ij}|\chi)$, $FA(\Theta''_{ij}|\chi)$ using Eq. (51).

Fig. 3. An algorithm for generating and evaluating appropriate frames of discernment.

and

$$\mathrm{Bel}(A) \leqslant \sum_{x \in A} p(x) \tag{54}$$

for all $A \subseteq \Theta$. AU was independently discovered by several authors about the same time [14–16]. For a recent overview, see [26].

Abellán et al. [27] suggested that AU could be disaggregated in separate measures of nonspecificity and scattering that generalize Hartley information [18] and Shannon entropy [17], respectively. Dubois and Prade [28] defined such a measure of nonspecificity as

$$I(m) = \sum_{A \in F} m \log_2 |A|, \tag{55}$$

where $F \subseteq 2^{\Theta}$ is the set of focal elements. From Eqs. (52) and (55) we may define a generalized Shannon entropy [27] as

$$GS(m) = AU(m) - I(m). \tag{56}$$

Disaggregation of AU into measures of nonspecificity and scattering is further discussed in a recent article by Bronevich and Klir [29]. They formulate axioms for uncertainty measures and extend measures of aggregated uncertainty to imprecise probabilities.

An algorithm for computing AU was initially found by Meyerowitz et al. [30]. This algorithm has a computational time complexity of $O(2^{|\Theta|})$ (see also [31] for implementation). Liu et al. [32] later developed the F-algorithm for calculating AU in order to reduce time complexity. Recently their presentation of the F-algorithm was corrected and further improved upon by Huynh and Nakamori (see Algorithm 3. The improved F-algorithm in [33, p. 208]). This improved F-algorithm restricts calculations to the unions of focal elements $U(F)$, where F is the set of focal elements. This step of the algorithm includes the leading term of the computational time complexity. From $|U(F)| \subseteq 2^{\Theta}$ they conclude that

Table 1
One piece of evidence m_1: Measures of frame appropriateness (FA), conflict and aggregated uncertainty (AU), do. normalized, generalized Shannon entropy (GS) and generalized Hartley information (GH) for Θ_1 (last row) and its six possible abridgements. R = 'Red', G = 'Green', B = 'Blue'.

Frame of discernment Θ_k	FA	Conflict	AU	$\frac{AU}{\log_2 \lvert\Theta_k\rvert}$	GS	GH
{R}	1	0	0	0	0	0
{G}	1	0	0	0	0	0
{B}	1	0	0	0	0	0
{R, G}	0.029	0	0.971	0.971	0.571	0.4
{R, B}	0.029	0	0.971	0.971	0.571	0.4
{R, B}	0	0	1	1	1	0
{R, G, B}	0	0	1.585	1	0.785	0.8

Table 2
Two pieces of evidence m_1 and m_2: Measures of frame appropriateness (FA), conflict and aggregated uncertainty (AU), do. normalized, generalized Shannon entropy (GS) and generalized Hartley information (GH) for Θ_1 (last row) and its 20 possible abridgements. R = 'Red', G = 'Green', B = 'Blue', F = 'Fast', VF = 'VeryFast'.

Frame of discernment Θ_k	FA	Conflict	AU	$\frac{AU}{\log_2 \lvert\Theta_k\rvert}$	GS	GH
{R} × {F}	1	0	0	0	0	0
{R} × {VF}	1	0	0	0	0	0
{R} × {F, VF}	0	0	1	1	0.2	0.8
{G} × {F}	1	0	0	0	0	0
{G} × {VF}	1	0	0	0	0	0
{G} × {F, VF}	0	0	1	1	0.2	0.8
{B} × {F}	1	0	0	0	0	0
{B} × {VF}	1	0	0	0	0	0
{B} × {F, VF}	0	0	1	1	0.2	0.8
{R, G} × {F}	0.029	0	0.971	0.971	0.571	0.4
{R, G} × {VF}	0.029	0	0.971	0.971	0.571	0.4
{R, G} × {F, VF}	0.014	0	1.971	0.986	0.771	1.2
{R, B} × {F}	0.029	0	0.971	0.971	0.571	0.4
{R, B} × {VF}	0.029	0	0.971	0.971	0.571	0.4
{R, B} × {F, VF}	0.014	0	1.971	0.986	0.771	1.2
{G, B} × {F}	0	0	1	1	1	0
{G, B} × {VF}	0	0	1	1	1	0
{G, B} × {F, VF}	0	0	2	1	1.2	0.8
{R, G, B} × {F}	0	0	1.585	1	0.785	0.8
{R, G, B} × {VF}	0	0	1.585	1	0.785	0.8
{R, G, B} × {F, VF}	0	0	2.585	1	0.985	1.6

$$|U(F)| \leqslant \min(|2^F|, |2^\Theta|), \tag{57}$$

which yields an improvement upon [30,32].

4. An algorithm for constructing an appropriate frame of discernment

Using the results of the preceding sections we develop an algorithm for constructing and evaluating all possible frames of discernment. This algorithm will first generate the possible frames using different partitions of the set of all cores (Step 1–3), Fig. 3. In Step 1 we generate the set of all cores $\{C_i\}$ from the belief functions. In Step 2 we generate the set of all partitions $\{\Omega_k\}$ of $\{C_i\}$. In Step 3 we use $\{\Omega_k\}$ to generate the set of all possible cross products $\{\Theta_k\}$. From these possible frames we generate abridgments (Step 4) and enlargements (Step 5). The frames are evaluated using the measure of frame appropriateness *FA*, Eq. (51) (Step 6). From the output of the algorithm the most appropriate frame that maximizes *FA* may be selected.

The frames of discernment $\left\{\Theta'_{ij}\right\}$ generated in step four may be generated recursively as long as all super sets have a conflict less than one.

Brute force implementation of *FA* has a computational time complexity of $O(|\chi|^{|\chi|}2^{|\Theta|})$. Implementing steps 2–4 in an iterative way may reduce the term $|\chi|^{|\chi|}$ of the time complexity.

Table 3
Three pieces of evidence m_1, m_2 and m_3: Measures of frame appropriateness (*FA*), conflict and aggregated uncertainty (*AU*), do. normalized, generalized Shannon entropy (*GS*) and generalized Hartley information (*GH*) for Θ_3 (last row) and its 44 possible abridgements. R = 'Red', G = 'Green', B = 'Blue', Bl = 'Black', F = 'Fast', VF = 'VeryFast'.

Frame of discernment Θ_k	*FA*	Conflict	*AU*	$\frac{AU}{\log_2 \lvert\Theta_k\rvert}$	*GS*	*GH*
{R} × {F}	1	0	0	0	0	0
{R} × {VF}	1	0	0	0	0	0
{R} × {F, VF}	0	0	1	1	0.2	0.8
{G} × {F}	0	1	0	0	0	0
{G} × {VF}	0	1	0	0	0	0
{G} × {F, VF}	0	1	0	0	0	0
{B} × {F}	0	1	0	0	0	0
{B} × {VF}	0	1	0	0	0	0
{B} × {F, VF}	0	1	0	0	0	0
{B1} × {F}	0	1	0	0	0	0
{B1} × {VF}	0	1	0	0	0	0
{B1} × {F, VF}	0	1	0	0	0	0
{R, G} × {F}	1	0	0	0	0	0
{R, G} × {VF}	1	0	0	0	0	0
{R, G} × {F, VF}	0.5	0	1	0.5	0.2	0.8
{R, B} × {F}	1	0	0	0	0	0
{R, B} × {VF}	1	0	0	0	0	0
{R, B} × {F, VF}	0.5	0	1	0.5	0.2	0.8
{R, B1} × {F}	1	0	0	0	0	0
{R, B1} × {VF}	1	0	0	0	0	0
{R, B1} × {F, VF}	0.5	0	1	0.5	0.2	0.8
{G, B} × {F}	0	1	0	0	0	0
{G, B} × {VF}	0	1	0	0	0	0
{G, B} × {F, VF}	0	1	0	0	0	0
{G, B1} × {F}	0	1	0	0	0	0
{G, B1} × {VF}	0	1	0	0	0	0
{G, B1} × {F, VF}	0	1	0	0	0	0
{B, B1} × {F}	0	1	0	0	0	0
{B, B1} × {VF}	0	1	0	0	0	0
{B, B1} × {F, VF}	0	1	0	0	0	0
{R, G, B} × {F}	1	0	0	0	0	0
{R, G, B} × {VF}	1	0	0	0	0	0
{R, G, B} × {F, VF}	0.613	0	1	0.387	0.2	0.8
{R, G, B1} × {F}	1	0	0	0	0	0
{R, G, B1} × {VF}	1	0	0	0	0	0
{R, G, B1} × {F, VF}	0.613	0	1	0.387	0.2	0.8
{R, B, B1} × {F}	1	0	0	0	0	0
{R, B, B1} × {VF}	1	0	0	0	0	0
{R, B, B1} × {F, VF}	0.613	0	1	0.387	0.2	0.8
{G, B, B1} × {F}	0	1	0	0	0	0
{G, B, B1} × {VF}	0	1	0	0	0	0
{G, B, B1} × {F, VF}	0	1	0	0	0	0
{R, G, B, B1} × {F}	1	0	0	0	0	0
{R, G, B, B1} × {VF}	1	0	0	0	0	0
{R, G, B, B1} × {F, VF}	0.667	0	1	0.333	0.2	0.8

If more belief functions arrive over time we must update the set of belief functions $\chi_{t+1} = \chi_t \cup \{m_j\}$ with the new belief functions $\{m_j\}$, Fig. 2, and recompute the evaluation of frame appropriateness, Fig. 3.

4.1. Revisit the example

Let us revisit the example and study the sequential process of frame construction as the first three pieces of evidence arrives.

Initially we have m_1, a piece of evidence about color. From this piece of evidence we can only construct one frame $\Theta_1 = \{R, G, B\}$ and its six possible abridgments, see Table 1.

With only one piece of evidence there is never any conflict, thus the minimization of entropy will favor frames of cardinality one. One possible extension is not normalizing the mass functions when some focal elements are eliminated entirely. This would introduce an internal conflict. If this approach was taken we would have a conflict for $\Theta'_1 = \{G\}$ and $\Theta'_1 = \{B\}$ and would favor $\Theta'_1 = \{R\}$.

When the second piece of evidence m_2 arrives we have focal elements that are non-exclusive forcing us to adopt a frame that is a cross product of the cores. It is here possible to construct two initial frames but only one Θ_1 abide type condition 2. There are 20 possible abridgments to Θ_1. As the combination of m_1 and m_2 are conflict free there will be no enlargements, see Table 2.

We notice six alternative frames of discernment with maximum frame appropriateness at this stage.

Finally, let us study the situation after the arrival of m_3. It is possible to abridge Θ_3 to a singleton subset {(Red, Fast)} with a frame appropriateness of 1 and support from the three belief functions of 1. There are also three possible frames with cardinality two, three frames with cardinality three and one frame with cardinality four, all of them with frame appropriateness of 1. These eight frames are all equally appropriate as they are all both conflict and entropy free, i.e., yielding maximal interesting interaction without any internal conflict. The other 37 possible frames all have a frame appropriateness of less than 1, see Table 3. As this example is conflict free for Θ_3 no enlargements of Θ_3 are generated in step 5 of Fig. 3.

It is obvious from Table 3 that small frames are favored as long as *AU* and the conflict do not grow. This is because *AU* is normalized to the interval [0, 1] by the logarithm of the size of the frame ($\log_2|\Theta_k|$). In particular, as long as all focal elements in the combination of all belief function share a common element for Θ_3, the frame can be reduced to that element as both conflict and entropy are zero for a singleton frame.

While the initial frame $\Theta_3 = \{R, G, B, B1\} \times \{F, VF\}$ is acceptable with *FA* = 0.667 it can be improved upon in several different ways, mainly by removing 'VeryFast' from the second cross product element. The frames of discernment with *FA* = 1 are the eight cross products of the (non-proper) subsets to the first cross product element that all contain 'Red'. In our case, all eight frames with *FA* = 1 yields the same result, i.e., $m_{1\oplus2\oplus3}$({(Red, Fast)}) = 1. This can be compared with Θ_3 where the support from the combination of the three belief functions is split among two focal elements {(Red, Fast), (Red, VeryFast)}. Thus, a more focused result is achieved by the frames with *FA* = 1.

5. Conclusions

We have developed a problem representation with which we can construct possible frames of discernment from incoming belief functions. These frames of discernment can be evaluated by a measure of frame appropriateness given the available evidence. Each frame is evaluated as to how well it yields interesting interaction among the available belief functions without exhibiting too much internal conflict.

With this methodology we are able to automate or semi-automate the most important part of probable reasoning: constructing the frame of discernment.

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